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THE
BENDIX
CORPORATION

GUIDANCE
SYSTEMS
DIVISION

TETERBORO
NEW JERSEY 07608

MODULAR DESIGN ATTITUDE CONTROL SYSTEM

FINAL REPORT
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PREPARED FOR:

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HUNTSVILLE, ALABAMA

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NOVEMBER 1, 1982

to

SEPTEMBER 30, 1983

APPROVED BY:

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ENGINEERING MANAGER
SYSTEMS DESIGN

PREPARED BY:

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TECHNICAL MANAGER

FOREWORD

This final report is submitted in accordance with "Scope of Work, Exhibit C" for Contract NAS8-33979. The study was directed from the Guidance Systems Division (GSD) of The Bendix Corporation. The engineering manager at this location was Mr. Joel Levinthal. Most of the analytical effort in support of this project was provided by Dr. Frederick Chichester, who wrote all sections of this report. The guidance of Dr. Henry B. Waites and Mr. Stan Carroll of MSFC during the course of this study is gratefully acknowledged.

ABSTRACT

The problem of applying modular attitude control to a rigid body - flexible suspension model of a flexible spacecraft with some state variables inaccessible was addressed by developing a sequence of single axis models and generating a series of reduced state linear observers of minimum order to reconstruct those scalar state variables that were inaccessible. The specific single axis models treated consisted of two, three and four rigid bodies, respectively, interconnected by a flexible shaft passing through the mass centers of the bodies. Reduced state linear observers of all orders up to one less than the total number of scalar state variables were generated for each of the three single axis models cited.

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SECTION 1

1.0 INTRODUCTION

This report is submitted in compliance with the Scope of Work under contract NAS8-33979. The period of performance covered by the contract is from October 15, 1982 to September 30, 1983. The submission and approval of this report constitute the successful completion of the "Exhibit C" portion of the contract.

This report is a sequel to four others, two of them previously submitted under a different contract number. The two prior reports, under a different contract number, references (1-1) and (1-2), were submitted in October 1978 and September, 1979 and covered the periods from July 27, 1977 to July 27, 1978 and from August 26, 1978 to August 26, 1979, respectively, in compliance with "Exhibit A" of contract NAS8-32660.

Two prior final reports were prepared under contract NAS-33979. Reference (1-3) was submitted on March 8, 1982 and covered the period from August 15, 1980 to October 15, 1981 in compliance with "Exhibit A" of the contract. Reference (1-4) was submitted on March 18, 1983 and covered the period from October 16, 1981 to October 31, 1982 in compliance with "Exhibit B".

1.1 OBJECTIVE

The sections that follow summarize the effort expended on the Modular Design Attitude Control System Study contract from November 1, 1982 to September 30, 1983. In prior applications of modular attitude control to rigid body-flexible suspension approximations of the rotational dynamics of prototype flexible spacecraft, it was assumed that all of the scalar state variables of the linearized models were accessible for measurement and/or control. Actual spacecraft to be controlled almost never satisfy such a broad condition. Therefore, the principal objective of the devel-

opment of modular attitude control, completed September 30, 1983, was the generation of a series of linear observers to support the application of control to state variable models of flexible spacecraft for which one or more state variables are inaccessible.

1.2 SCOPE

Study effort was concentrated in two main areas:

A. Development of a series of single axis state variable models of flexible spacecraft to be utilized in the comparison of different approaches to the development of modular attitude control systems. These models consisted of two, three or four rigid bodies serially connected by a flexible suspension in such a way that motion was restricted to rotation about a common axis through the mass centers of the bodies.

B. Generation of reduced state linear observers for each single axis model developed in Task A corresponding to various numbers and distributions of inaccessible state variables following the approaches presented in Luenberger (1-5), (1-6), (1-7), and Sage (1-8).

1.3 GENERAL

This report is comprised of five sections. Sections 2 and 3 describe the development of the two and three body single axis state variable models, respectively, of a prototype flexible spacecraft and the generation of the minimum order reduced state linear observers for the reconstruction of inaccessible scalar state variables of these models. Section 4 portrays the expansion of the three body single axis state variable model to a four body model as an example of the effects of adding another mass to an existing model and describes the generation of the minimum order reduced state linear observers for various numbers and distributions of inaccessible scalar state variables of the four body model. Section 5 lists a number of conclusions and recommendations drawn from generation of linear

observers for the series of single axis state variable models described above. References are listed at the end of each section.

The original RFQ requested that the International System of units (designated as SI) be used in the program and in any reporting. Torques, moments, angular momentum, moments of inertia and distances, however, are stated in English units since this was the system of units used in presenting all of the vehicle data in the RFQ.

1.4 REFERENCES

1-1 Guidance Systems Division, The Bendix Corporation, "Space Construction Base Control System", Final Report, Contract NAS8-32660 for George C. Marshall Space Flight Center, October 27, 1978.

1-2 Guidance Systems Division, The Bendix Corporation, "Space Construction Base Control System", Final Report, Contract NAS8-32660 for George C. Marshall Space Flight Center, September 1, 1979.

1-3 Guidance Systems Division, The Bendix Corporation, "Modular Design Attitude Control System", Final Report, Contract NAS8-33979 for George C. Marshall Space Flight Center, March 8, 1982.

1-4 Guidance Systems Division, The Bendix Corporation, "Modular Design Attitude Control System", Final Report, Contract NAS8-33979 for George C. Marshall Space Flight Center, March 18, 1983.

1-5 Luenberger, D.G., "Determining the state of a Linear System with Observers of Low Dynamic Order", Ph.D. dissertion, Stanford University, 1963.

1-6 Luenberger, D.G., "Observers for Multivariable Systems", IEEE Transactions on Automatic Control, Vol. AC-11, No. 2, April 1966, pp. 190-197.

1-7 Luenberger, D.G., "An Introduction to Observers", IEEE Transactions on Automatic Control, Vol. AC-16, No. 6, December 1971, pp. 596-602.

1-8 Sage, A.P., Optimum Systems Control. Englewood Cliffs, N.J.: Prentice-Hall, Inc. 1968, pp. 306-312.

SECTION 2

2.0 DEVELOPMENT OF TWO BODY SINGLE AXIS MODEL AND ITS REDUCED STATE LINEAR OBSERVERS

2.1 MODEL EQUATIONS

The state variable form of the two body single axis model of a flexible spacecraft shown in Figure 2-1, assuming m of its four scalar state variables are accessible, may be expressed as follows.

$$\underline{\dot{x}} = A\underline{x} + B\underline{u} \tag{2-1}$$

$$\underline{y} = C\underline{x} \tag{2-2}$$

where:

$\underline{x} = (x_1, x_2, x_3, x_4)^T = (\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)^T$ = state vector

$\underline{u} = (u_1, u_2)^T = \left(\frac{T_1}{I_1} \quad \frac{T_2}{I_2}\right)^T$ = control vector

$\underline{y} = (y_1, \ldots, y_m)^T$ = vector of measured or observed states (m = 1,2,3)

A = 4×4 state vector coefficient matrix

B = 4×r control vector coefficient matrix (r = 1 or 2)

C = m×4 measurement or observation vector coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_{23} & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 1 \\ a_{41} & 0 & -a_{41} & 0 \end{bmatrix} \tag{2-3}$$

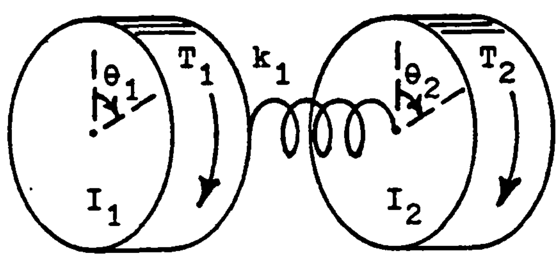


FIGURE 2-1

TWO BODY SINGLE AXIS MODEL

$$a_{23} = \frac{k_1}{I_1} \tag{2-4}$$

$$a_{41} = \frac{k_1}{I_2} \tag{2-5}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{2-6}$$

The block diagram corresponding to this model is depicted in Figure 2-2.

2.2 REDUCED STATE LINEAR OBSERVERS

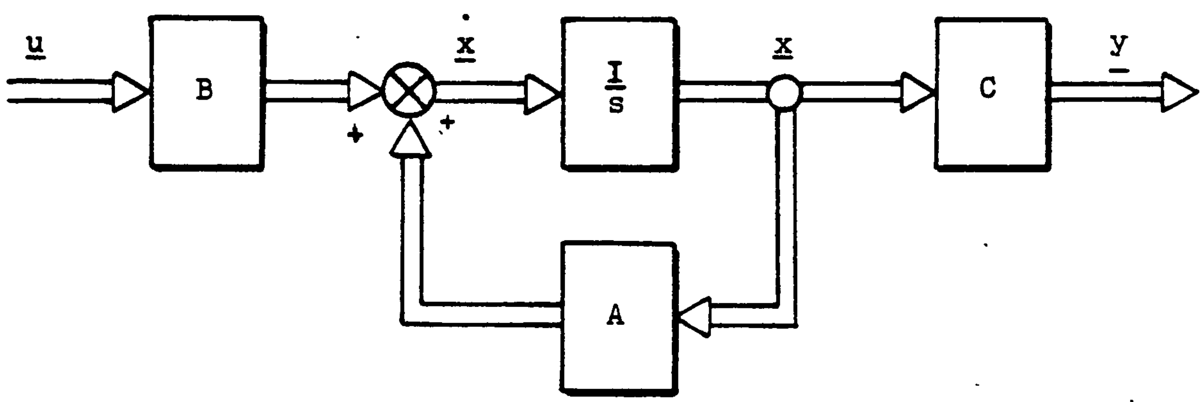
2.2.1 Introduction

The minimum order (number of scalar state variables) of a reduced state linear observer required to reconstruct the 4-m inaccessible scalar states of the two body single axis model represented by equations (2-1) through (2-6) is p = 4-m. This reconstruction was accomplished for a given state variable model in three main stages.

1) Synthesizing a linear observer of minimum required order (p).

2) Defining a synthesized variable corresponding to each of the inaccessible state variables of the given state variable model.

3) Expressing each synthesized variable as a function of the state variables of the reduced state observer and the accessible state variables of the given state variable model.



State and Observation Equations:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

$$\underline{y} = C\underline{x}$$

FIGURE 2-2

LINEARIZED STATE VARIABLE MODEL OF THE SYSTEM TO BE CONTROLLED

The equations for the reduced state observers corresponding to the state variable model of equations (2-1) through (2-6) are the following.

$$\dot{\underline{z}} = D\underline{z} + E\underline{u} + G\underline{y} \tag{2-7}$$

$$\underline{z} = T\underline{x} \tag{2-8}$$

where:

$$\left.\begin{array}{l} D = p \times p \text{ observer coefficient matrix (assumed diagonal)} \\ E = TB = p \times 4 \text{ observer control vector coefficient matrix} \\ G = p \times m \text{ observer vector of observed states coefficient matrix} \\ T = p \times 4 \text{ observer weighting matrix} \end{array}\right. \tag{2-9}$$

The corresponding block diagram appears in Figure 2-3.

2.2.2 Observer Synthesis Equations

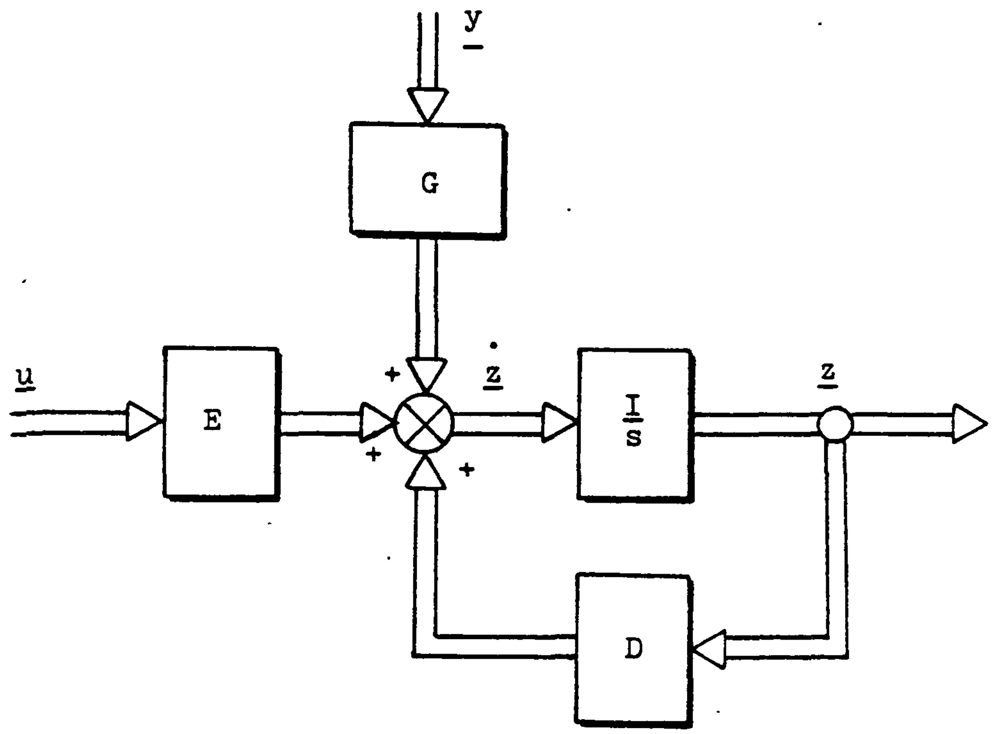
The equations for synthesizing the reduced state linear observers, based on those appearing in Luenberger (2-1), (2-2) and (2-3) and Sage (2-4), were written in the following form.

$$TA - DT = F \tag{2-10}$$

$$F = GC, \tag{2-11}$$

For

$$D = \begin{bmatrix} d_{11} & & \bigcirc \\ & \ddots & \\ \bigcirc & & d_{p,p} \end{bmatrix} \tag{2-12}$$



Observer Equations:

$$\dot{\underline{z}} = D\underline{z} + G\underline{y} + E\underline{u}$$

Since $G\underline{y} = GC\underline{x} = F\underline{x}$,

$$\dot{\underline{z}} = D\underline{z} + F\underline{x} + E\underline{u}$$

FIGURE 2-3

LINEAR OBSERVER CORRESPONDING TO

LINEARIZED STATE VARIABLE MODEL OF FIGURE 2-2

$$
T = \begin{bmatrix} t_{11} & t_{12} & t_{12} & t_{14} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ t_{p,1} & t_{p,2} & t_{p,3} & t_{p,4} \end{bmatrix} \tag{2-13}
$$

$$
F = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ f_{p,1} & f_{p,2} & f_{p,3} & f_{p,4} \end{bmatrix} \tag{2-14}
$$

and the form of the A matrix given in equation (2-3) the observer synthesis equations reduce to the following general forms.

$$
\begin{aligned}
t_{i2} &= \frac{(\Delta_{i2})_{1,1}(f_{i1}+d_{ii}f_{i2}) - (\Delta_{i2})_{2,1}(f_{i3}+d_{ii}f_{i4})}{\Delta_{i2}} \\
t_{i4} &= \frac{-(\Delta_{i2})_{1,2}(f_{i1}+d_{ii}f_{i2}) + (\Delta_{i2})_{2,2}(f_{i3}+d_{ii}f_{i4})}{\Delta_{i2}} \\
t_{i1} &= d_{ii}t_{i2} + f_{i2} \qquad\qquad i = 1, \ldots, p \\
t_{i3} &= d_{ii}t_{i4} + f_{i4}
\end{aligned} \tag{2-15}
$$

where

$$\Delta_{i2} = \begin{vmatrix} -(d_{ii}^2+a_{23}) & a_{41} \\ a_{23} & -(d_{ii}^2+a_{41}) \end{vmatrix} \tag{2-16}$$

$$= d_{ii}^{2}(d_{ii}^{2}+a_{23}+a_{41})$$

$(\Delta_{i2})_{i,j} = \Delta_{i2}$ without the elements of the ith row and jth column.

Inaccessibility of a state variable in the model equations (2-1), (2-2) is reflected by a corresponding null column in the observation matrix, C, and a corresponding null column in the F matrix as implied by equation (2-11). For the generation of reduced order observers for the two body model the number of inaccessible state variables can be 1, 2 or 3.

2.3 FIRST ORDER OBSERVERS (p = 1)

A first order linear observer corresponds to inaccessiblity of one of the four scalar state variables of the two body model. Therefore, the total number of first order linear observers that can be synthesized for the two body model is given by:

$$C_1^4 = 4$$

The observer equation then reduces to:

$$\dot{z} = dz + E\underline{u} + G\underline{y}, \tag{2-17}$$

the F and T matrices reduce to:

$$F = [f_1 \; f_2 \; f_3 \; f_4]$$

$$T = [t_1 \; t_2 \; t_3 \; t_4] \qquad (2\text{-}18)$$

and the observer synthesis equations, equation set (2-15) and equation (2-16), reduce to the following forms.

$$t_2 = \frac{(\Delta_2)_{1,1}(f_1+df_2) - (\Delta_2)_{2,1}(f_3+df_4)}{\Delta_2}$$

$$t_4 = \frac{-(\Delta_2)_{1,2}(f_1+df_2) + (\Delta_2)_{2,2}(f_3+df_4)}{\Delta_2}$$

$$t_1 = dt_2 + f_2.$$

$$t_3 = dt_4 + f_4 \qquad (2\text{-}19)$$

$$\Delta_2 = \begin{vmatrix} -(d^2+a_{23}) & a_{41} \\ a_{23} & -(d^2+a_{41}) \end{vmatrix} = d^2(d^2+ a_{23}+ a_{41})$$

$(\Delta_2)_{i,j} = \Delta_2$ without the elements of the ith row and jth column

Since this case corresponds to inaccessibility of <u>one</u> state variable, one of the f_i $(i = 1, \ldots, 4) = 0$.

Example

Suppose x_4, the scalar state representing the angular rate of body 2, is inaccessible.

Then it is assumed that:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2-21}$$

for which:

$$F = [\ f_1\ f_2\ f_3\ 0\ ] \tag{2-22}$$

and T is of the form shown in equation (2-18).

From equations (2-10), (2-21) and (2-22),

$$G = [\ f_1\ f_2\ f_3\ ] \tag{2-23}$$

and from equations (2-4), (2-9) and (2-18)

$$E = [\ t_2\ t_4\ ] \tag{2-24}$$

This equation corresponds to r = 2, control torques applied to both bodies. For control torque applied only to body 1,

$$E = [\ t_2\ 0\ ] \tag{2-25}$$

and for control torque applied only to body 2,

$$E = [\ 0\ t_4\ ] \tag{2-26}$$

The equations for determining the elements of the T matrix reduce to the following forms.

$$t_2 = \frac{(\Delta_2)_{1,1}(f_1+df_2) - (\Delta_2)_{2,1}f_3}{\Delta_2}$$

$$t_4 = \frac{-(\Delta_2)_{1,2}(f_1+df_2) + (\Delta_2)_{2,2}f_3}{\Delta_2}$$

$$t_1 = dt_2 + f_2 \qquad (2\text{-}27)$$

$$t_3 = dt_4$$

From equations (2-8) and (2-18),

$$z = t_1x_1 + t_2x_2 + t_3x_3 + t_4\hat{x}_4 \qquad (2\text{-}28)$$

where $\hat{x}_4'$ = the synthesized x_4.

Solving for $\hat{x}_4$ yields:

$$\hat{x}_4 = \frac{1}{t_4}\left[z - \sum_{i=1}^{3} t_ix_i\right] \qquad (2\text{-}29)$$

For inaccessibility of each of the remaining three scalar state variables the equations for determining t_i, set (2-19) and equations (2-8) and (2-21) through (2-29) are appropriately modified.

2.4 SECOND ORDER OBSERVERS (p = 2)

The equation for a linear observer of order two corresponds to two of the four scalar state variables being inaccessible. It is represented here as equation (2-7). The total number of second order observers that can be synthesized for the two body model is given by:

$$C_2^4 = \frac{4!}{2!2!} = 6$$

If the observer coefficient matrix is assumed to be diagonal in this case, it appears as follows:

$$D = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix} \tag{2-30}$$

Since the observer is of order two,

$$\underline{z} = (z_1, z_2)^T \tag{2-31}$$

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \end{bmatrix} \tag{2-32}$$

and

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \end{bmatrix} \tag{2-33}$$

The specific forms of the equations for generating the elements of T depend upon which two of the scalar states are inaccessible. For each inaccessible state the corresponding columns in the C and F matrices are null.

Example

Corresponding to the angular position and rate, respectively, of body 2, suppose that the scalar states x_3 and x_4 are inaccessible. Then the equations for generating the elements of the T matrix assume the following forms.

$$t_{i2} = \frac{(\Delta_{i2})_{1,1}(f_{i1}+d_{ii}f_{i2})}{\Delta_{i2}}$$

$$i = 1,\ 2$$

$$t_{i4} = \frac{-(\Delta_{i2})_{1,2}(f_{i1}+d_{ii}f_{i2})}{\Delta_{i2}} \qquad (2\text{-}34)$$

$$t_{i1} = d_{ii}t_2 + f_{i2}$$

$$t_{i3} = d_{ii}t_{i4}$$

$$\Delta_{i2} = \begin{vmatrix} -(d_{ii}^2+a_{23}) & a_{41} \\ a_{23} & -(d_{ii}^2+a_{41}) \end{vmatrix} = d_{ii}^2(d_{ii}^2+a_{23}+a_{41}) \qquad (2\text{-}35)$$

$(\Delta_{i2})_{i,j} = \Delta_{i2}$ without elements of ith row and jth column

From equation (2-8),

$$\begin{bmatrix} t_{13} & t_{14} \\ t_{23} & t_{24} \end{bmatrix} \begin{bmatrix} \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} = \begin{bmatrix} z_1 - t_{11}x_1 - t_{12}x_2 \\ z_2 - t_{21}x_1 - t_{22}x_2 \end{bmatrix} \tag{2-36}$$

where $\hat{x}_3$ and $\hat{x}_4$ are synthesized state variables.

Let $\Delta_2 = \begin{vmatrix} t_{13} & t_{14} \\ t_{23} & t_{24} \end{vmatrix} = t_{13}\, t_{24} - t_{14}\, t_{23} \neq 0$

$(\Delta_2)_{i,j} = \Delta_2$ without elements of ith row and jth column

$$\hat{x}_3 = \frac{(\Delta_2)_{1,1}(z_1 - t_{11}x_1 - t_{12}x_2) - (\Delta_2)_{2,1}(z_2 - t_{21}x_1 - t_{22}x_2)}{\Delta_2} \tag{2-37}$$

$$\hat{x}_4 = \frac{-(\Delta_2)_{1,2}(z_1 - t_{11}x_1 - t_{12}x_2) + (\Delta_2)_{2,2}(z_2 - t_{21}x_1 - t_{22}x_2)}{\Delta_2} \tag{2-38}$$

For x_3 and x_4 inaccessible, it is assumed that:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{2-39}$$

$$F = \begin{bmatrix} f_{11} & f_{12} & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 \end{bmatrix} \tag{2-40}$$

From F = GC,

$$G = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \tag{2-41}$$

From E = TB,

$$E = \begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \end{bmatrix} \quad \text{for } r = 2 \text{ (control torques applied to both bodies)} \tag{2-42}$$

$$E = \begin{bmatrix} t_{12} & 0 \\ t_{22} & 0 \end{bmatrix} \quad \text{for control restricted to body 1} \tag{2-43}$$

$$E = \begin{bmatrix} 0 & t_{14} \\ 0 & t_{24} \end{bmatrix} \quad \text{for control restricted to body 2} \tag{2-44}$$

2.5 THIRD ORDER OBSERVERS (p = 3)

The equation for the linear observer of order one less than the system's dimension corresponds to three of the four scalar state variables being inaccessible. It is represented here as equation (2-7). The total number of third order linear observers that can be synthesized for the two body model is given by:

$$C_3^4 = \frac{4!}{3!2!} = 4$$

If the observer coefficient matrix is assumed to be diagonal in this case it appears as follows,

$$D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \tag{2-45}$$

Since the observer is of order 3,

$$\underline{z} = (z_1, z_2, z_3)^T \tag{2-46}$$

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \end{bmatrix} \tag{2-47}$$

and

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \end{bmatrix} \tag{2-48}$$

The specific forms of the equations for generating the elements of T depend upon which three of the scalar states are inaccessible. For each inaccessible state the corresponding columns in the C and F matrices are null.

Example

Suppose the scalar states, x_2, x_3 and x_4, representing the angular rate of body 1 and the angular position and rate of body 2, are inaccessible. Then the equations for generating the elements of the T matrix assume the following form since $f_{i2} = f_{i3} = f_{i4} = 0$ for $i = 1, 2, 3$.

$$t_{i2} = \frac{(\Delta_{i2})_{1,1} f_{i1}}{\Delta_{i2}}$$

$$i = 1, 2, 3$$

$$t_{i4} = \frac{-(\Delta_{i2})_{1,2} f_{i1}}{\Delta_{i2}} \tag{2-49}$$

$$t_{i1} = d_{ii} t_{i2}$$

$$t_{i3} = d_{ii} t_{i4}$$

$$\Delta_{i2} = \begin{vmatrix} -(d_{ii}^2 + a_{23}) & a_{41} \\ a_{23} & -(d_{ii}^2 + a_{41}) \end{vmatrix} = d_{ii}^2 (d_{ii}^2 + a_{23} + a_{41}) \tag{2-50}$$

$(\Delta_{i2})_{i,j} = \Delta_{12}$ without elements of ith row and jth column

From equation (2-8).

$$\begin{bmatrix} t_{12} & t_{13} & t_{14} \\ t_{22} & t_{23} & t_{24} \\ t_{32} & t_{33} & t_{34} \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} = \begin{bmatrix} z_1 - t_{11} x_1 \\ z_2 - t_{21} x_1 \\ z_3 - t_{31} x_1 \end{bmatrix} \tag{2-51}$$

where $\hat{x}_2$, $\hat{x}_3$ and $\hat{x}_4$ are synthesized state variables.

$$\text{Let } \Delta_3 = \begin{vmatrix} t_{12} & t_{13} & t_{14} \\ t_{22} & t_{23} & t_{24} \\ t_{32} & t_{33} & t_{34} \end{vmatrix} \neq 0$$

where

$(\Delta_3)_{i,j} = \Delta_3$ without elements of ith row and jth column

$$\hat{x}_{j+1} = \frac{\sum_{i=1}^{3} (-1)^{i+j} (\Delta_3)_{i,j} (z_j - x_1)}{\Delta_3} \qquad j = 1, 2, 3 \qquad (2\text{-}52)$$

For x_2, x_3 and x_4 inaccessible, it is assumed that:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \qquad (2\text{-}53)$$

$$F = \begin{bmatrix} f_{11} & 0 & 0 & 0 \\ f_{21} & 0 & 0 & 0 \\ f_{31} & 0 & 0 & 0 \end{bmatrix} \qquad (2\text{-}54)$$

From $F = GC$,

$$G = \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \end{bmatrix} \qquad (2\text{-}55)$$

From $E = TB$,

$$E = \begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \\ t_{32} & t_{34} \end{bmatrix} \quad \begin{array}{l} \text{for } r = 2 \\ \text{(control torques applied to both bodies)} \end{array} \qquad (2\text{-}56)$$

2.6 REFERENCES

2-1 Luenberger, D.G., "Determining the state of a Linear System with Observers of Low Dynamic Order", Ph.D. dissertion, Stanford University, 1963.

2-2 Luenberger, D.G., "Observers for Multivariable Systems", IEEE Transactions on Automatic Control, Vol. AC-11, No. 2, April 1966, pp. 190-197.

2-3 Luenberger, D.G., "An Introduction to Observers", IEEE Transactions on Automatic Control, Vol. AC-16, No. 6, December 1971, pp. 596-602.

2-4 Sage, A.P., Optimum Systems Control. Englewood Cliffs, N.J.: Prentice-Hall, Inc. 1968, pp. 306-312.

SECTION 3

3.0 DEVELOPMENT OF THREE BODY SINGLE AXIS MODEL AND ITS REDUCED STATE LINEAR OBSERVERS

3.1 MODEL EQUATIONS

The state variable form of the three body single axis model of a flexible spacecraft shown in Figure 3-1 may be expressed as follows:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{3-1}$$

$$\underline{y} = C\underline{x} \tag{3-2}$$

where:

$\underline{x} = (x_1, x_2, x_3, x_4, x_5, x_6)^T = (\Theta_1, \dot{\Theta}_1, \Theta_2, \dot{\Theta}_2, \Theta_3, \dot{\Theta}_3)^T$

= state vector

$\underline{u} = (u_1, \ldots, u_r)^T = \left(\frac{T_1}{I_1}, \ldots, \frac{T_r}{I_r}\right)^T \quad (r = 1,\ 2 \text{ or } 3)$

$\underline{y} = (y_1, \ldots y_m)^T =$ vector of measured or observed states

C = observation matrix of dimensions $m \times 6$, $m = 1, 2, \ldots, 5$ (Minimum dimension of reduced order observer required = $6-m$).

Partitioning of this model by rigid body results in the following forms for its coefficient matrices.

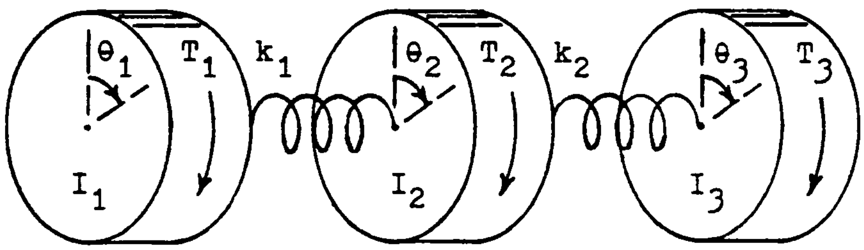


FIGURE 3-1

THREE BODY SINGLE AXIS MODEL

$$
A = \left[\begin{array}{cc|cc|cc}
0 & 1 & 0 & 0 & 0 & 0 \\
-a_{23} & 0 & a_{23} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 \\
a_{41} & 0 & a_{43} & 0 & a_{45} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & a_{63} & 0 & -a_{63} & 0
\end{array}\right] \tag{3-3}
$$

$$
\begin{aligned}
a_{23} &= \frac{k_1}{I_1} \\
a_{41} &= \frac{k_1}{I_2} \\
a_{45} &= \frac{k_2}{I_2} \\
a_{43} &= -(a_{41}+a_{45}) \\
a_{63} &= \frac{k_2}{I_3}
\end{aligned} \tag{3-4}
$$

$$
B = \left[\begin{array}{c|c|c}
0 & 0 & 0 \\
1 & 0 & 0 \\
\hline
0 & 0 & 0 \\
0 & 1 & 0 \\
\hline
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right] \quad \text{for } r = 3 \text{ (control torques applied to all three bodies)} \tag{3-5}
$$

The block diagram corresponding to this model is shown in Figure 2-2.

3.2 REDUCED STATE LINEAR OBSERVERS

3.2.1 Introduction

For the three body single axis model represented by equations (3-1) through (3-5), the minimum order of a reduced state linear observer required to generate the inaccessible states is $p = 6-m$ ($m = 1, 2, .., 5$). All of the reduced state linear observers for the three body model may be written in the form represented by equations (2-7) and (2-8) where, in this case, the observer coefficient matrix, D, is assumed to be diagonal and of dimensions $p \times p$. The corresponding observer weighting matrix is of the following form.

$$T = \begin{bmatrix} t_{11} & \cdot & \cdot & \cdot & \cdot & t_{16} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ t_{p,1} & \cdot & \cdot & \cdot & \cdot & t_{p,6} \end{bmatrix} \tag{3-6}$$

From equations (2-9), (3-5) and (3-6).

$$E = \begin{bmatrix} t_{12} & t_{14} & t_{16} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ t_{p,2} & t_{p,4} & t_{p,6} \end{bmatrix} \quad \text{for } r = 3 \text{ (control torques applied to all 3 bodies).} \tag{3-7}$$

$$T = \begin{bmatrix} f_{11} & \cdot & \cdot & \cdot & \cdot & f_{16} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ f_{p,1} & \cdot & \cdot & \cdot & \cdot & f_{p,6} \end{bmatrix} \tag{3-8}$$

The corresponding observer block diagram appears in Figure 2-3.

3.2.2 Observer Synthesis Equations

From Luenburger (3-1), (3-2), (3-3) and Sage (3-4) the equations for synthesizing the reduced state linear observers for the three body single axis model represented by equations (3-1) through (3-5) are given by equations (2-10) and (2-11). With coefficient matrices of the forms listed in 3.2.1 this set of observer sythesis equations reduces to the following forms.

$$t_{i,2k} = \frac{(-1)^{i+k}(\Delta_{i3})_{1,k}(f_{i}+d_{ii}f_{i4}) + (-1)^{i+k+1}(\Delta_{i3})_{2,k}(f_{i3}+d_{ii}f_{i4})}{\Delta_{i3}}$$

$$+ \frac{(-1)^{i+k+2}(\Delta_{i3})_{3,k}(f_{i5}+d_{ii}f_{i6})}{\Delta_{i3}} \tag{3-9}$$

$i = 1,2,\ldots,p$

$k = 1,2,3$

$t_{i,2k-1} = d_{ii}t_{i,2k} + f_{i,2k}$

where t_{ij} are elements of the T matrix

and

$$\Delta_{i3} = \begin{vmatrix} -(d_{ii}^2+a_{23}) & a_{41} & 0 \\ a_{23} & -(d_{ii}^2+a_{41}+a_{45}) & a_{63} \\ 0 & a_{45} & -(d_{ii}^2+a_{63}) \end{vmatrix} \tag{3-10}$$

$$= -(d_{ii}{}^2 + a_{23})(\Delta_{i3})_{1,1} - a_{23}(\Delta_{i3})_{2,1}$$

where

$(\Delta_{i3})_{i,j} = \Delta_{i3}$ without the elements of the ith row and jth column

Inaccessibility of a scalar state variable in equation set (3-1), (3-2) is reflected by a corresponding null column in the observation matrix, C and, as implied by equation (2-11), in the F matrix for the generation of reduced order observers for the three body model the number of inaccessible scalar states can be 1, 2, 3, 4 or 5.

3.3 FIRST ORDER OBSERVERS (p = 1)

A first order observer is required when only one of the six scalar state variables of the three body model is inaccessible. Hence, the total number of first order observers that can be generated for the three body model is given by:

$$C_1^6 = 6$$

The first order form of the linear observer equation is:

$$\dot{z} = dz + E\underline{u} + G\underline{y} \tag{3-11}$$

The F and T matrices associated with a first order observer for the three body model then reduce to the following row forms.

$$F = [f_1 \ f_2 \ f_3 \ f_4 \ f_5 \ f_6]^T \tag{3-12}$$

$$T = [t_1 \ t_2 \ t_3 \ t_4 \ t_5 \ t_6]^T \tag{3-13}$$

The observer synthesis equations are then given by equation set (3-9) and equation (3-10) with i = 1.

Since a first order observer corresponds to one of the scalar state variables being inaccessible, one of the f_i $(i=1, 2, 3, 4, 5, 6) = 0$.

Example

Suppose that the scalar state representing the angular rate of body 3, x_6, is inaccessible. Then $f_6 = 0$ and the observer synthesis equations reduce to the following forms.

$$t_{2k} = \frac{(-1)^{k+1}(\Delta_3)_{1,k}(f_1+df_2) + (-1)^{k+2}(\Delta_3)_{2,k}(f_3+df_4)}{\Delta_3}$$

$$+ \frac{(-1)^{k+3}(\Delta_3)_{3,k}f_5}{\Delta_3} \qquad k = 1, 2, 3 \tag{3-14}$$

$$t_{2k-1} = dt_{2k} + f_{2k} \qquad k = 1, 2$$

$$t_5 = dt_6$$

From equation (2-8) the synthesized scalar state, $\hat{x}_6$, is expressed in terms of the observer variable, z_1, and the accessible scalar state variables as follows.

$$\hat{x}_6 = \frac{1}{t_6}\left[z - \sum_{i=1}^{5} t_i x_i\right] \tag{3-15}$$

In this case, it is assumed that:

$$C = \left[\begin{array}{c|c} I_5 & \begin{matrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{matrix} \end{array}\right] \tag{3-16}$$

where I_5 = 5×5 identity matrix

From F = GC,

$$G = [\ f_1\ f_2\ f_3\ f_4\ f_5\ ] \tag{3-17}$$

From E = TB,

$$E = [\ t_2\ \ t_4\ \ t_6\ ] \text{ for } r = 3 \tag{3-18}$$

(control torques applied to all three bodies)

$$E = [\ t_2\ \ t_4\ \ 0\ ] \text{ for control applied to bodies 1 and 2} \tag{3-19}$$

$$E = [\ t_2\ \ 0\ \ 0\ ] \text{ for control applied to body 1} \tag{3-20}$$

$$E = [\ t_2\ \ 0\ \ t_6\ ] \text{ for control applied to bodies 1 and 3} \tag{3-21}$$

3.4 OBSERVERS OF INTERMEDIATE ORDER (p = 2, 3 or 4)

In the cases in which an intermediate number of the six scalar states of the three body single axis model is inaccessible the minimum order of the reduced state linear observer required to reconstruct these inaccessible states is given by p. In each case the number of null columns in the measurement or observation matrix, C, and the F matrix also is equal to p. The general forms of the E, F and T matrices are given in equations (3-6), (3-7) and (3-8) for p = 2, 3 or 4 where p represents the number of inaccessible state variables of the model. If any two of the six scalar states of the three body model are inaccessible, then the total number of second order observers that can be generated for this model is given by:

$$n_2 = C_2^6 = \frac{6 \cdot 5}{2} = 15 \tag{3-22}$$

In general, if any p of the six scalar states are inaccessible, the number of observers of order p that can be generated for the three body single axis model is:

$$n_p = C_p^6 = \frac{6!}{p!(6-p)!} \tag{3-23}$$

With the assumption of a diagonal D matrix the observer synthesis equations are given by equation set (3-9) and equation (3-10) where $i = 1, \ldots, p$.

Example

Suppose the scalar states, x_5 and x_6, corresponding to the angular position and rate, respectively, of body 3, are inaccessible. Then $f_{i5} = f_{i6} = 0$ for $i = 1, 2$ and the observer synthesis equations reduce to the following forms for the required second order observer.

$$t_{i,2k} = \frac{(-1)^{i+k}(\Delta_{i3})_{1,k}(f_{i1}+d_{ii}f_{i2}) + (-1)^{i+k+1}(\Delta_{i3})_{2,k}(f_{i3}+d_{ii}f_{i4})}{\Delta_3} \tag{3-24}$$

$$k = 1, 2, 3 \qquad i = 1, 2$$

$$t_{i,2k-1} = d_{ii}t_{i,2k} + f_{i,2k} \qquad k = 1, 2$$

Where Δ_{i3} is expanded in equation (3-10) and $(\Delta_{i3})_{i,j}$ is Δ_{i3} without the elements of the ith row and jth column.

From equation (2-8) the synthesized scalar states, $\hat{x}_5$ and $\hat{x}_6$, are expressed in terms of the observer variables, z_1 and z_2, and the accessible state variables as follows.

$$\hat{x}_5 = \frac{(\Delta_2)_{1,1}(z_1 - \sum_{j=1}^{4} t_{1j}x_j) - (\Delta_2)_{2,1}(z_2 - \sum_{j=1}^{4} t_{2j}x_j)}{\Delta_2} \tag{3-25}$$

$$\hat{x}_6 = \frac{-(\Delta_2)_{1,2}(z_1 - \sum_{j=1}^{4} t_{1j}x_j) + (\Delta_2)_{2,2}(z_2 - \sum_{j=1}^{4} t_{2j}x_j)}{\Delta_2} \tag{3-26}$$

where $(\Delta_2)_{i,j} = \Delta_2$ without the elements of the ith row and jth column.

For x_5 and x_6 inaccessible, it is assumed that:

$$C = \left[\begin{array}{c|cc} & 0 & 0 \\ I_4 & \cdot & \cdot \\ & \cdot & \cdot \\ & 0 & 0 \end{array}\right] \tag{3-27}$$

where I_4 = 4×4 identity matrix

From F = GC,

$$G = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \end{bmatrix} \tag{3-28}$$

From E = TB,

$$E = \begin{bmatrix} t_{12} & t_{14} & t_{16} \\ t_{22} & t_{24} & t_{26} \end{bmatrix} \text{ for } r = 3 \text{ (control torques applied to all three bodies)} \qquad (3\text{-}29)$$

$$E = \begin{bmatrix} t_{12} & t_{14} & 0 \\ t_{22} & t_{24} & 0 \end{bmatrix} \text{ for control applied to bodies 1 and 2.} \qquad (3\text{-}30)$$

$$E = \begin{bmatrix} t_{12} & 0 & t_{16} \\ t_{22} & 0 & t_{26} \end{bmatrix} \text{ for control applied to bodies 1 and 3.} \qquad (3\text{-}31)$$

$$E = \begin{bmatrix} t_{12} & 0 & 0 \\ t_{22} & 0 & 0 \end{bmatrix} \text{ for control restricted to body 1.} \qquad (3\text{-}32)$$

3.5 FIFTH ORDER OBSERVERS (p = 5)

An observer of at least order five is required when any five of the six scalar state variables of the three body models are inaccessible. Therefore, the total number of fifth order observers that can be generated for the three body model is expressed by the following.

$$n_5 = C_5^6 = 6 \qquad (3\text{-}33)$$

The observer synthesis equations are given in equation set (3-9) and equation (3-10) with $i = 1, 2, \ldots, 5$.

Since a fifth order observer corresponds to five of the six scalar states being inaccessible, $f_{1j} = f_{2j} = f_{3j} = f_{4j} = f_{5j} = 0$ for five of the six values of the subscript, j.

Example

Suppose that the scalar states, x_2, x_3, x_4, x_5 and x_6, representing the angular rate of body 1 and the angular displacements and rates of bodies 2 and 3 are inaccessible. Then $f_{i2} = f_{i3} = f_{i4} = f_{i5} = f_{i6} = 0$ for $i = 1, 2, \ldots, 5$ and the observer synthesis equations reduce to the following forms.

$$t_{i,2k} = \frac{(-1)^{i+k}(\Delta_{i3})_{1,k} f_{i1}}{\Delta_{i3}} \qquad \begin{matrix} i = 1, 2, .., 5 \\ k = 1, 2, 3 \end{matrix} \tag{3-34}$$

$$t_{i,2k-1} = d_{ii} t_{i,2k}$$

where Δ_{i3} is expanded in equation (3-10) and $(\Delta_{i3})_{i,j}$ is Δ_{i3} without the elements of the ith row and jth column.

The synthesized scalar state variables, $\hat{x}_2$, $\hat{x}_3$, $\hat{x}_4$, $\hat{x}_5$ and $\hat{x}_6$ are expressed in terms of the observer scalar variables. z_1, $z_2 \ldots z_5$, and the accessible state variables, using equation (2-8) as follows:

$$\hat{x}_{j+1} = \frac{\sum_{i=1}^{5} (-1)^{i+j} (\Delta_5)_{i,j} (z_i - x_1)}{\Delta_5} \qquad j = 1, 2, .., 5 \tag{3-35}$$

$$\Delta_5 = \begin{vmatrix} t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \\ t_{22} & \cdot & \cdot & \cdot & t_{26} \\ t_{32} & \cdot & \cdot & \cdot & t_{36} \\ t_{42} & \cdot & \cdot & \cdot & t_{46} \\ t_{52} & t_{53} & t_{54} & t_{55} & t_{46} \end{vmatrix}$$

$$= t_{12}(\Delta_5)_{1,1} - t_{22}(\Delta_5)_{2,1} + t_{32}(\Delta_5)_{3,1} - t_{42}(\Delta_5)_{4,1} + t_{52}(\Delta_5)_{5,1} \tag{3-36}$$

where $(\Delta_5)_{i,\,j} = \Delta_5$ without the elements of the ith row and jth column.

For only x_1 accessible, it is assumed that:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3-37}$$

From F = GC,

$$G = \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{41} \\ f_{51} \end{bmatrix} \tag{3-38}$$

From E = TB,

$$E = \begin{bmatrix} t_{12} & t_{14} & t_{16} \\ t_{22} & \cdot & t_{26} \\ t_{32} & \cdot & t_{36} \\ t_{42} & \cdot & t_{46} \\ t_{52} & t_{54} & t_{56} \end{bmatrix} \quad \text{for } r = 3 \text{ (control torques applied to all three bodies).} \tag{3-39}$$

3.6 REFERENCES

3-1 Luenberger, D.G., "Determining the state of a Linear System with Observers of Low Dynamic Order", Ph.D. dissertion, Stanford University, 1963.

3-2 Luenberger, D.G., "Observers for Multivariable Systems", IEEE Transactions on Automatic Control, Vol. AC-11, No. 2, April 1966, pp. 190-197.

3-3 Luenberger, D.G., "An Introduction to Observers", IEEE Transactions on Automatic Control, Vol. AC-16, No. 6, December 1971, pp. 596-602.

3-4 Sage, A.P., Optimum Systems Control. Englewood Cliffs, N.J.: Prentice-Hall, Inc. 1968, pp. 306-312.

SECTION 4

4.0 DEVELOPMENT OF FOUR BODY SINGLE AXIS MODEL AND ITS REDUCED STATE LINEAR OBSERVERS

4.1 MODEL EQUATIONS

The state variable form of the four body single axis model of a flexible spacecraft depicted in Figure 4-1 was written in the following form.

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{4-1}$$

$$\underline{y} = C\underline{x} \tag{4-2}$$

where:

$$\underline{x} = (x_1, \; . \; . \; . \; . \; . \; . \; ., \; x_8)^T$$
$$= (\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, \theta_3, \dot{\theta}_3, \theta_4, \dot{\theta}_4)^T$$

$$\underline{u} = (u_1, \; . \; ., \; u_r)^T = \left(\frac{T_1}{I_1}, \; . \; ., \; \frac{T_r}{I_r}\right)^T \qquad (r = 1, 2, 3 \text{ or } 4)$$

$$\underline{y} = (y_1, \; . \; . \; . \; ., \; y_m)^T \qquad (m = 1, 2, 3, 4, 5, 6 \text{ or } 7)$$

C = m×8 measurement or observation matrix

Partitioning of this model by rigid body yields the following forms for its coefficient matrices:

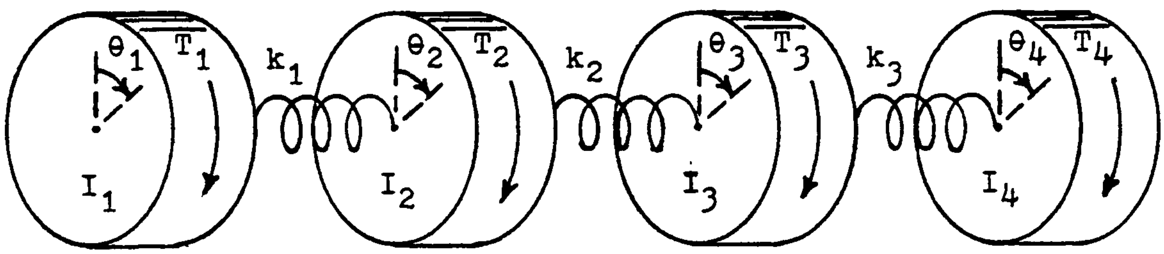


FIGURE 4-1

FOUR BODY SINGLE AXIS MODEL

$$
A = \left[\begin{array}{cc|cc|cc|cc}
0 & 1 & 0 & 0 & \multicolumn{2}{c|}{[0]} & \multicolumn{2}{c}{[0]} \\
-a_{23} & 0 & a_{23} & 0 & & & & \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & \multicolumn{2}{c}{[0]} \\
a_{41} & 0 & a_{43} & 0 & a_{45} & 0 & & \\
\hline
\multicolumn{2}{c|}{[0]} & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & a_{63} & 0 & a_{65} & 0 & a_{67} & 0 \\
\hline
\multicolumn{2}{c|}{[0]} & \multicolumn{2}{c|}{[0]} & 0 & 0 & 0 & 1 \\
 & & & & a_{85} & 0 & -a_{85} & 0
\end{array}\right] \tag{4-3}
$$

$$
[0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

$$
B = \left[\begin{array}{c|c|c|c}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right] \quad \text{for } r = 4 \tag{4-4}
$$

$$a_{23} = \frac{k_1}{I_1}$$

$$a_{41} = \frac{k_1}{I_2}, \qquad a_{45} = \frac{k_2}{I_2}, \qquad a_{43} = -(a_{41}+a_{45})$$

$$a_{63} = \frac{k_2}{I_3}, \qquad a_{67} = \frac{k_3}{I_3}, \qquad a_{65} = -(a_{63}+a_{67}) \tag{4-5}$$

$$a_{85} = \frac{k_3}{I_4}$$

The corresponding block diagram appears in Figure 2-2.

4.2 COMPARISON OF THREE BODY AND FOUR BODY MODELS

4.2.1 Introduction

In order to evaluate the effects of adding another rotational mass to a single axis model the three and four body models partitioned by rigid body were compared. More specifically the partitioned forms of the coefficient matrices appearing in the state variable form of the model, equations (4-1) and (4-2) were compared.

4.2.2 Comparison of "A" Matrices of Three and Four Body Models

The three body and four body A matrices partitioned by rigid body were presented in equations (3-3) and (4-3), respectively. The result of superimposing the three body A matrix upon the four body A matrix appears as follows.

$$
A = \left[\begin{array}{cc:cc|cc|cc}
0 & 1 & 0 & 0 & \multicolumn{2}{c|}{} & \multicolumn{2}{c}{} \\
-\frac{k_1}{I_1} & 0 & \frac{k_1}{I_1} & 0 & \multicolumn{2}{c|}{[0]} & \multicolumn{2}{c}{[0]} \\
\hdashline
0 & 0 & 0 & 1 & 0 & 0 & \multicolumn{2}{c}{} \\
\frac{k_1}{I_2} & 0 & -\frac{k_1+k_2}{I_2} & 0 & \frac{k_2}{I_2} & 0 & \multicolumn{2}{c}{[0]} \\
\hdashline
\multicolumn{2}{c:}{} & 0 & 0 & 0 & 1 & 0 & 0 \\
\multicolumn{2}{c:}{[0]} & \frac{k_2}{I_3} & 0 & -\frac{k_2}{I_3}-\frac{k_3}{I_3} & 0 & \frac{k_3}{I_4} & 0 \\
\hline
\multicolumn{2}{c:}{} & \multicolumn{2}{c|}{} & 0 & 0 & 0 & 1 \\
\multicolumn{2}{c:}{[0]} & \multicolumn{2}{c|}{[0]} & \frac{k_3}{I_4} & 0 & -\frac{k_3}{I_4} & 0
\end{array}\right] \tag{4-6}
$$

$$
[0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

On the right hand side of the above equation the last two columns of the matrix, which appear due to the addition of the fourth body, are partitioned from the remainder of the matrix by solid lines. The addition of the fourth body also results in the appearance of another term at the intersection of the sixth row and fifth column. Careful review of the overall pattern of non-zero elements in this matrix partitioned by rigid body implies that each addition of a rigid body would result in the addition of corresponding elements with respect to the two rows and columns added.

For example, addition of a fifth rigid body to the four body model would have the following effects.

1) Addition of the term, $-\frac{k_4}{I_4}$, to the element at the intersection of the eighth row and seventh column.

2) Addition of a ninth and tenth row and a ninth and tenth column containing the following elements:

   - "1" at the intersection of the ninth row and tenth column
   - $\frac{"k_4"}{I_5}$ at the intersection of the eighth row and ninth column and at the tenth row and seventh column.
   - $-\frac{"k_4"}{I_5}$ at the intersection of the tenth row and ninth column.
   - "0" for each of the remaining elements.

4.2.3 Comparison of "B" Matrices of Three and Four Body Models

Generalized forms of the three body and four body B matrices were superimposed for the case in which no control torques were applied to the fourth body. The result has the following form.

$$B = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ b_{11} & b_{12} & b_{13} \\ \hline 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} \\ \hline 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad (4\text{-}7)$$

It is evident that in this case the addition of the fourth body without control actuators results in the addition of two null rows to the B matrix.

4.2.4 Comparison of "C" Matrices of Three and Four Body Models

The specific form of the C matrix associated with the three body model depends upon which of the six scalar state variables of the model is inaccessible. If the state variable, x_6, representing the angular rate of body 3 is inaccessible then the C matrix can be assumed to be of the form given in equation (3-16). Partitioning this matrix by rigid body produces the following form

$$C = \left[\begin{array}{c|c|cc} I_2 & [0] & 0 & 0 \\ \hline [0] & I_2 & 0 & 0 \\ \hline 0 \quad 0 & 0 \quad 0 & 1 & 0 \end{array}\right] \tag{3-16}$$

where:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad [0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Addition of a fourth body to the model for which neither its angular position nor angular rate is accessible would result in the addition of two null columns to the matrix on the right hand side of equation (3-16). For each accessible scalar state variable associated with the fourth body another row would be added to the C matrix with a "1" element in a position corresponding to that variable. Hence, if one of the two state variables associated with the fourth body were accessible, addition of the body would add one null column and a non-null row and column with a "1" at their common intersection. If both of the state variables associated with the fourth body are accessible, the addition of this body to the model

adds two non-null columns and two non-null rows to the matrix on the right hand side of equation (3-16). The superposition of the three and four body C matrices partitioned by rigid body yields the following.

$$C = \left[\begin{array}{c|c|c|c} I_2 & [0] & [0] & [0] \\ \hline [0] & I_2 & [0] & [0] \\ \hline 0\ 0 & 0\ 0 & 1\ 0 & 0\ 0 \\ \hline [0] & [0] & [0] & I_2 \end{array}\right] \tag{4-8}$$

where:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad [0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

4.3 REDUCED STATE LINEAR OBSERVERS

4.3.1 Introduction

The minimum order of a reduced state linear observer required to reconstruct the 8-m inaccessible scalar state variables of the four body single axis model of a flexible spacecraft represented by equations (4-1) through (4-5) is $p = 8-m$ where $m = 1, 2, 3, 4, 5, 6$ or 7. All of the reduced state linear observers for this four body model may be written in the form of equations (2-7) and (2-8) under the assumption that the observer coefficient matrix, D, is diagonal and of dimensions $p \times p$. The corresponding observer weighting matrix is of the following form.

$$T = \begin{bmatrix} t_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & t_{18} \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ t_{p,1} & \cdot & \cdot & \cdot & \cdot & \cdot & t_{p,8} \end{bmatrix} \tag{4-9}$$

From equations (2-9), (4-4) and (4-9),

$$E = \begin{bmatrix} t_{12} & t_{14} & t_{16} & t_{18} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ t_{p,2} & t_{p,4} & t_{p,6} & t_{p,8} \end{bmatrix} \tag{4-10}$$

$$F = \begin{bmatrix} f_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & f_{18} \\ \cdot & & & & & & & & \cdot \\ \cdot & & & & & & & & \cdot \\ f_{p,1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & f_{p,8} \end{bmatrix} \tag{4-11}$$

The corresponding observer block diagram appears in Figure 2-3.

4.3.2 Observer Synthesis Equations

From Luenberger (4-1), (4-2) and (4-3) and Sage (4-4) the equations for synthesizing the reduced state linear observers for the four body single axis model represented by equations (4-1) through (4-5) are given by equations (2-10) and (2-11). With coëfficient matrices of the form listed in 4.3.1 this set of observer synthesis equations reduces to the following.

$$t_{i,2k} = \frac{(-1)^{i+k}(\Delta_{i4})_{1,k}(f_{i1}+d_{ii}f_{i2}) + (-1)^{i+k+1}(\Delta_{i4})_{2,k}(f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}$$

$$+ \frac{(-1)^{i+k+2}(\Delta_{i4})_{3,k}(f_{i5}+d_{ii}f_{i6}) + (-1)^{i+k+3}(\Delta_{i4})_{4,k}(f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}} \tag{4-12}$$

$i = 1, 2, \ldots, p$

$k = 1, 2, 3, 4$

$$t_{1,2k-1} = d_{ii}t_{i,1,2k} + f_{i,2k}$$

where t_{ij} are elements of the T matrix.

and

$$\Delta_{i4} = \begin{vmatrix} -(d_{ii}^2+a_{23}) & a_{41} & 0 & 0 \\ a_{23} & -(d_{ii}^2+a_{41}+a_{45}) & a_{63} & 0 \\ 0 & a_{45} & -(d_{ii}^2+a_{63}+a_{67}) & a_{85} \\ 0 & 0 & a_{67} & -(d_{ii}^2+a_{85}) \end{vmatrix} \tag{4-13}$$

$$= -(d_{ii}^2+a_{23})(\Delta_{i4})_{1,1} - a_{23}(\Delta_{i4})_{2,1}$$

$(\Delta_{i4})_{i,\ j} = \Delta_{i4}$ without the elements of the ith row and jth column.

Inaccessibility of a scalar state variable in the model equations (4-1), (4-2) is reflected by a corresponding null column in the C and F matrices as implied in equation (2-11). For the generation of reduced state observers for the four body model the number of inaccessible state variables, p, can be 1, 2, 3, 4, 5, 6 or 7.

4.4 FIRST ORDER OBSERVERS (p = 1)

An observer of order at least one is requred when only <u>one</u> of the eight scalar state variables of the four body model is inaccessible. Therefore, the total number of first order observers that can be generated for the four body model is given by:

$C_1^8 = 8$

The first order form of the linear observer equation is as follows:

$$\dot{z} = dz + E\underline{u} + G\underline{y} \tag{4-14}$$

The F and T matrices associated with a first order observer for the four body model then reduce to the following row forms.

$$F = [\ f_1\ f_2\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ f_8\ ] \tag{4-15}$$

$$T = [\ t_1\ t_2\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ t_8\ ] \tag{4-16}$$

The observer synthesis equations are then of the form of equation set (4-12) and equation (4-13) with $i = 1$.

Since a first order observer corresponds to one of the scalar state variables being inaccessible, <u>one</u> of the f_i $(i = 1, 2, .., 8) = 0$.

<u>Example</u>

Suppose the scalar state representing the angular rate of body 4, x_8, is inaccessible. Then $f_8 = 0$ and the observer synthesis equations reduce to the following forms.

$$t_{2k} = \frac{(-1)^{k+1}(\Delta_4)_{1,k}(f_1+df_2) + (-1)^{k+2}(\Delta_4)_{2,k}(f_3+df_4)}{\Delta_4}$$

$$+ \frac{(-1)^{i+k+3}(\Delta_4)_{3,k}(f+df_6) + (-1)^{k+4}(\Delta_4)_{4,k}f_7}{\Delta_{i4}}$$

$$k = 1,\ 2,\ 3,\ 4$$

$$\tag{4-17}$$

$$t_{2k-1} = dt_{2k} + f_{2k} \qquad k = 1,\ 2,\ 3$$

$$t_7 = dt_8$$

$$\Delta_4 = \begin{vmatrix} -(d^2+a_{23}) & a_{41} & 0 & 0 \\ a_{23} & -(d^2+a_{41}+a_{45}) & a_{63} & 0 \\ 0 & a_{45} & -(d^2+a_6+a_{67}) & a_{85} \\ 0 & 0 & a_{67} & -(d^2+a_{85}) \end{vmatrix}$$

$$= -(d^2+a_{23})(\Delta_4)_{1,1} - a_{23}(\Delta_4)_{2,1} \tag{4-18}$$

where $(\Delta_4)_{i,\,j} = \Delta_4$ without the elements of the ith row and jth column from equation (2-8), the synthesized scalar state, $\hat{x}_8$, is expressed in terms of the scalar observer variable, z, and the accessible scalar state variables as follows.

$$\hat{x}_8 = \frac{1}{t_8}\left[z - \sum_{i=1}^{7} t_i x_i\right] \tag{4-19}$$

For x_8 inaccessible, it is assumed that:

$$C = \left[\begin{array}{c|c} & 0 \\ & \cdot \\ & \cdot \\ I_7 & \cdot \\ & \cdot \\ & \cdot \\ & 0 \end{array}\right], \tag{4-20}$$

where I_7 = 7×7 identity matrix,

From $F = GC$,

$$G = [\ f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7\ ] \tag{4-21}$$

From $E = TB$,

$$E = [\ t_2\ t_4\ t_6\ t_8\ ] \quad \text{for } r = 4 \text{ (control torques on all 4 bodies)} \tag{4-22}$$

4.5 OBSERVERS OF INTERMEDIATE ORDER ($p = 2, 3, 4, 5$ or 6)

For those cases in which an intermediate number of the eight scalar states of the four body single axis model is inaccessible, the minimum order of the reduced state linear observer required to reconstruct these inaccessible states is given by p. In each case the number of null columns in the measurement or observation matrix, C, and the F matrix also is equal to p. The general forms of the E, F and T matrices are given in equations (4-9), (4-10) and (4-11) for $p = 2, 3, 4, 5$ or 6 where p represents the number of inaccessible scalar state variables of the model. If any p of the eight scalar state variables of the four body models are inaccessible, then the total number of observers of order p that can be generated for this model is given by:

$$n_p = C_p^8 = \frac{8!}{p!(8-p)!} \tag{4-23}$$

<u>Example</u>

Suppose the scalar states, x_7 and x_8, which represent the angular position and rate of body 4, are inaccessible. Then $f_{i7} = f_{i8} = 0$ for $i = 1, 2$ and the observer synthesis equations reduce to the following forms.

$$t_{i,2k} = \frac{(-1)^{i+k}(\Delta_{i4})_{1,k}(f_{i1}+d_{ii}f_{i2}) + (-1)^{i+k+1}(\Delta_{i4})_{2,k}(f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}$$

$$+ \frac{(-1)^{i+k+2}(\Delta_{i4})_{3,k}(f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}} \tag{4-24}$$

$i = 1, 2$

$k = 1, 2, 3, 4$

$t_{i,2k-1} = d_{ii}t_{i,2k} + f_{i,2k}$ $\qquad k = 1, 2, 3$

$t_{i,7} = d_{ii}t_{i,8}$

Where Δ_{14} is expanded in equation (4-13) and $(\Delta_{14})_{i,\, j}$ is Δ_{14} without the elements of the ith row and jth column.

From equation (2-8) the synthesized scalar states, $\hat{x}_7$ and $\hat{x}_8$, are expressed in terms of the scalar observer variables, z and z_2 and the accessible scalar state variables as follows.

$$\hat{x}_7 = \frac{\sum_{i=1}^{2}(-1)^{i+1}(\Delta_2)_{i,1}(z_i - \sum_{j=1}^{6} t_{ij}x_j)}{\Delta_2} \tag{4-25}$$

$$\hat{x}_8 = \frac{\sum_{i=1}^{2}(-1)^{i+1}(\Delta_2)_{i,2}(z_i - \sum_{j=1}^{6} t_{ij}x_j)}{\Delta_2} \tag{4-26}$$

for

$$\Delta_2 = \begin{vmatrix} t_{17} & t_{18} \\ t_{27} & t_{28} \end{vmatrix} = t_{17}t_{28} - t_{18}t_{27} \neq 0 \tag{4-27}$$

Where $(\Delta_2)_{i,j} = \Delta_2$ without the elements of the ith row and jth column.

For x_7 and x_8 inaccessible, it is assumed that:

$$C = \left[\begin{array}{c|cc} & 0 & 0 \\ & \cdot & \cdot \\ & \cdot & \cdot \\ I_6 & \cdot & \cdot \\ & \cdot & \cdot \\ & 0 & 0 \end{array}\right] \tag{4-28}$$

Since F = GC,

$$G = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \end{bmatrix} \tag{4-29}$$

From E = TB,

$$E = \begin{bmatrix} t_{12} & t_{14} & t_{16} & t_{18} \\ t_{22} & t_{24} & t_{26} & t_{28} \end{bmatrix} \quad \text{for } r = 4 \text{ (control applied to all four bodies} \tag{4-30}$$

$$E = \begin{bmatrix} t_{12} & t_{14} & t_{16} & 0 \\ t_{22} & t_{24} & t_{26} & 0 \end{bmatrix} \quad \text{(for control torques applied to bodies 1, 2 and 3)} \tag{4-31}$$

$$E = \begin{bmatrix} t_{12} & t_{14} & 0 & 0 \\ t_{22} & t_{24} & 0 & 0 \end{bmatrix} \quad \text{(for control torques applied to bodies 1 and 2)} \tag{4-32}$$

$$E = \begin{bmatrix} t_{12} & 0 & 0 & 0 \\ t_{22} & 0 & 0 & 0 \end{bmatrix} \quad \text{(for control torques applied to body 1)} \tag{4-33}$$

4.6 SEVENTH ORDER OBSERVERS (p = 7)

When any seven of the eight scalar state variables of the four body models are inaccessible, a linear observer of at least order seven is required. The total number of seventh order observers that can be generated for the four body models may be expressed as follows.

$$C_7^8 = 8 \tag{4-34}$$

The observer synthesis equations are as presented in equation set (4-12) and equation (4-13) with $i = 1, 2, \ldots, 7$. Since a seventh order observer corresponds to seven of the scalar states being inaccessible, $f_{1j} = f_{2j} = \ldots = f_{7j} = 0$ for seven of the eight values of the subscript, j.

Example

Suppose only the scalar state variable representing the angular position of body 1, x_1, is accessible. Then the remaining scalar states, x_2, $x_3 \ldots, x_8$ are inaccessible, $f_{i2} = f_{i3} = \ldots = f_{i8} = 0$ for $i = 1, 2, 3, 4, 5, 6$ and 7 and the observer synthesis equations reduce to the following forms.

$$t_{i,2k} = \frac{(-1)^{i+k}(\Delta_{i4})_{1,k} f_{i1}}{\Delta_{i4}} \qquad k = 1, 2, 3, 4 \qquad i = 1, 2, \ldots, 7 \tag{4-35}$$

$$t_{i,2k-1} = d_{i1} t_{i,2k} \qquad k = 1, 2, 3, 4$$

Where Δ_{i4} is expanded in equation (4-13) and $(\Delta_{i4})_{i,j}$ is Δ_{i4} without the elements in the ith row and jth column.

The synthesized scalar state variables, $\hat{x}_2$ through $\hat{x}_8$, are expressed in terms of the observer variables. z_1 through z_7, and the accessible state variable, x_1, by utilizing equation (2-8) in the following form.

$$\hat{x}_{k+1} = \frac{\sum_{i=1}^{7}(-1)^{i+1}(\Delta_7)_{i,k}(z_i - t_{i1}x_1)}{\Delta_7} \qquad k = 1, 2, \ldots, 7 \tag{4-36}$$

$$\Delta_7 = \begin{vmatrix} t_{12} & t_{13} & t_{14} & t_{15} & t_{16} & t_{17} & t_{18} \\ t_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & t_{28} \\ t_{32} & \cdot & \cdot & \cdot & \cdot & \cdot & t_{38} \\ t_{42} & \cdot & \cdot & \cdot & \cdot & \cdot & t_{48} \\ t_{52} & \cdot & \cdot & \cdot & \cdot & \cdot & t_{58} \\ t_{62} & \cdot & \cdot & \cdot & \cdot & \cdot & t_{68} \\ t_{72} & t_{73} & t_{74} & t_{75} & t_{76} & t_{77} & t_{78} \end{vmatrix}$$

$$= t_{12}(\Delta_7)_{1,1} - t_{22}(\Delta_7)_{2,1} + t_{32}(\Delta_7)_{3,1} - t_{42}(\Delta_7)_{4,1} + t_{52}(\Delta_7)_{5,1} - t_{62}(\Delta_7)_{6,1} + t_{72}(\Delta_7)_{7,1} \qquad (4\text{-}37)$$

Where $(\Delta_7)_{i,j} = \Delta_7$, without the elements of the ith row and jth column.

For only x_1 accessible, it is assumed that:

$$C = [\, 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \,] \qquad (4\text{-}38)$$

From $F = GC$,

$$G = \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{41} \\ f_{51} \\ f_{61} \\ f_{71} \end{bmatrix}$$

From E = TB,

$$E = \begin{bmatrix} t_{12} & t_{14} & t_{16} & t_{18} \\ t_{22} & \cdot & \cdot & t_{28} \\ t_{32} & \cdot & \cdot & t_{38} \\ t_{42} & \cdot & \cdot & t_{48} \\ t_{52} & \cdot & \cdot & t_{58} \\ t_{62} & \cdot & \cdot & t_{68} \\ t_{72} & t_{74} & t_{76} & t_{78} \end{bmatrix} \quad \text{for } r = 4 \text{ (control torques applied to all four bodies)} \qquad (4\text{-}40)$$

4.7 REFERENCES

4-1 Luenberger, D.G., "Determining the state of a Linear System with Observers of Low Dynamic Order", Ph.D. dissertion, Stanford University, 1963.

4-2 Luenberger, D.G., "Observers for Multivariable Systems", IEEE Transactions on Automatic Control, Vol. AC-11, No. 2, April 1966, pp. 190-197.

4-3 Luenberger, D.G., "An Introduction to Observers", IEEE Transactions on Automatic Control, Vol. AC-16, No. 6, December 1971, pp. 596-602.

4-4 Sage, A.P., Optimum Systems Control. Englewood Cliffs, N.J.: Prentice-Hall, Inc. 1968, pp. 306-312.

SECTION 5

5.0 CONCLUSIONS AND RECOMMENDATIONS

During the period covered by this report three single axis models of a prototype flexible spacecraft were written in state variable form and a series of reduced state linear observers of various orders was generated for each single axis model. Each of the single axis models developed consisted of two or more bodies restricted to rotational motion about a common axis through their centers of mass. A distinct spring coefficient was associated with each interconnection between the masses of each model. The resulting linear models were written in state variable form and later partitioned by rigid body. This partitioning was done to facilitate the expansion of the models by sequential addition of rigid bodies. The single axis model development treated in this report commenced with a two body model and progressed to three and four body models. Each of these models involved a transformation from the vector of scalar state variables to a vector of measured or observed scalar states to represent the physical situation in which one or more of the scalar states was inaccessible.

For each combination of single axis state variable model and inaccessible scalar state(s) a reduced state linear observer was generated to reconstruct those scalar states that were inaccessible. This was done because the application of linear quadratic regulator (LQR) and closely related time domain approaches to attitude control utilize all or nearly all of the scalar states of the model of the spacecraft to be controlled.

5.1 CONCLUSIONS

The following conclusions were drawn mainly from the development of the two body, three body and four body single axis models with inaccessible scalar state variables of a prototype flexible spacecraft and the generation of the corresponding linear observers of minimum order required to reconstruct these inaccessible scalar states.

1. Comparison of the coefficient matrices of single axis state variable models in ascending order of numbers of rigid bodies revealed that successive A, B and C matrices form patterns that permit easy extension to models involving larger numbers of rigid bodies. This comparison was facilitated by partitioning each model by rigid body.

2. For the single axis state variable models treated in this report an r body model has 2r scalar states consisting of angular position and rate for each rigid body.

3. The minimum order required for a reduced state linear observer to reconstruct p inaccessible scalar states of a single axis state variable model with a total of n scalar states is p where p = 1, 2, ..., n-1.

4. Under assumption of a diagonal D matrix in the reduced state linear observer of minimum order to reconstruct the inaccessible scalar states of a given single axis state variable model the observer synthesis equations reduce to a relatively simple form depending upon the number of rigid bodies involved.

5. Comparison of the observer synthesis equations for the minimum order linear observers corresponding to single axis models involving successively higher numbers of rigid bodies revealed that the observer synthesis equations could be expanded easily to accomodate the addition of another rigid body to the single axis model.

6. For the case in which any one of the n scalar states of the single axis model is inaccessible the following statements apply.

   a. The minimum required order of the corresponding linear observer is one.

   b. The total number of first order observers that can be generated is n.

c. The E, F, G and T matrices associated with the first order observer each contain a single row.

d. Each linear equation in the set of observer synthesis equations expresses an element of the row matrix, T, in terms of d and the corresponding element of the row matrix, F, of the first order observer and elements of the A matrix of the single axis model.

e. Solution for the one synthesized scalar state of the single axis model in terms of the one scalar state of the first order observer and the n-1 accessible scalar states of the single axis model involves no matrix inversions.

7. For the case in which any p of the n scalar states of the single axis model are inaccessible the following statements apply where p = 2, 3, ..., n-2.

   a. The minimum required order of the corresponding linear observer is p.

   b. The total number of observers of order p that can be generated is given by:

   $$C_p^n = \frac{n!}{p!(n-p)!}$$

   c. The E matrix is of dimensions $p \times (n/2)$.

   d. The F and T matrices are each of dimensions $p \times n$

   e. The G matrix is of dimensions $p \times (n-p)$

   f. The observer synthesis equations consist of sets of p linear equations of the same form with each equation within one of these sets expressing an element in a given column of the T matrix in

terms of the elements in the corresponding columns of the D and F matrices of the observer and elements of the A matrix of the single axis model.

g. Solution for each of the p synthesized scalar states of the single axis model in terms of the p observer scalar states and the n-p accessible scalar states of the single axis model requires inversion of a matrix of dimensions $p \times p$.

8. For the case in which any n-1 of the n scalar states of the single axis model are inaccessible the following statements apply.

a. The minimum required order of the corresponding linear observer is n-1.

b. The total number of observers of order n-1 that can be generated is n which is the same as for the first order observer.

c. The E matrix is of dimensions $(n-1) \times (n/2)$.

d. The F and T matrices each are of dimensions $(n-1) \times n$.

e. The G matrix is of dimensions $(n-1) \times 1$ (vector of dimension n-1).

f. The observer synthesis equations consist of sets of n-1 linear equations of the same form with each equation within one of these sets expressing an element in a given column of the T matrix in terms of the given elements in the corresponding columns of the D and F matrices of the observer and elements of the A matrix of the single axis model.

g. Solution for each of the n-1 synthesized scalar states of the single axis model in terms of the n-1 observer scalar state of the single axis model requires inversion of a matrix of dimensions $(n-1)\times(n-1)$.

9. A multi-axis rigid body-flexible joint model of a flexible spacecraft with weak coupling between its axes might be represented approximately under certain conditions by the corresponding number of single axis models.

5.2 RECOMMENDATIONS

The following directions are suggested for future study in the application of attitude control to state variable models of flexible spacecraft for which one or more scalar states are inaccessible.

1. The modular control techniques developed for the attitude control of models of flexible spacecraft for which all scalar state variables are accessible should be modified for application to the series of single axis models and their associated reduced state linear observers developed in the work treated in this report.

2. Selected combinations of single axis model and its associated linear observer and modular attitude control system should be simulated on a digital computer to support investigation of effects of changes in the following single axis model and observer characteristics.

   a. Ratios between the masses (rotational inertias) of bodies comprising the single axis model

   b. Magnitudes of spring and damping coefficients at the interfaces between the rigid bodies of the single axis model

c. Distribution and weighting of control torques applied to the rigid bodies of the single axis model

d. Number and distribution of inaccessible scalar state variables of the single axis model

e. Magnitudes of the elements of the D and E or F matrices of the reduced state observer

f. Presence of non-zero off-diagonal elements in the D matrix of the reduced state observer

3. Effects of damping in single axis models upon the synthesis of the corresponding reduced state linear observers should be evaluated.

4. The generation of reduced state observers to reconstruct inaccessible scalar states of a model of a flexible spacecraft should be extended to the three axis five body model of a prototype flexible spacecraft developed earlier. This extension could be accomplished in the following sequence of steps.

   a. Extend the four body single axis model to a five body single axis model that could represent the axis of the linearized three axis five body model that was found to be decoupled from the other two axes of the model in earlier work.

   b. Extend the generation of reduced state linear observers for the four body single axis model with one or more inaccessible scalar states to the corresponding five body model.

c. Extend the generation of linear observers for a sequence of single axis models treated in this report to a sequence of two axis models involving interaxial coupling culminating in a five body two axis model that could represent the two coupled axes of the three axis five body model of a prototype flexible spacecraft developed previously.

5. The application of modular techniques to the attitude control of selected combinations of a single axis model and its corresponding reduced state linear observer should be extended to the combination of the single axis and two axis five body models representing the prototype flexible spacecraft and the corresponding reduced state observers.

6. The combination of single axis and two axis five body models and their linear observers and modular attitude control systems should be simulated on a digital computer.

7. Coefficients representing the sensitivity of the scalar states to parameters of the combination of single axis and two axis five body models and their linear observers and modular attitude control systems should be developed.